

# ALMOST AUTOMORPHIC DELAYED DIFFERENTIAL EQUATIONS AND LASOTA-WAZEWSKA MODEL

ANÍBAL CORONEL, CHRISTOPHER MAULÉN, MANUEL PINTO, DANIEL SEPULVEDA

**ABSTRACT.** Existence of almost automorphic solutions for abstract delayed differential equations is established. Using ergodicity, exponential dichotomy and Bi-almost automorphy on the homogeneous part, sufficient conditions for the existence and uniqueness of almost automorphic solutions are given.

## 1. INTRODUCTION

The development of the theory of almost periodic type functions has been strongly stimulated by problems arising in differential equations, stability theory, dynamical systems and many other areas of science. Nowadays, there exist also a wide range of applications starting from the basic mathematical models based on linear ordinary differential equations, including nonlinear linear ordinary differential equations, differential equations in Banach space and also partial differential equations. Moreover, there exist several related concepts which arise as generalizations of the almost periodic concept. For instance the notions of almost automorphic, asymptotically almost periodic, asymptotically almost automorphic and pseudo almost periodic. Since there are plenty of results in literature, let us just quote, for their applications in engineering and life science, for example asymptotically almost periodic functions [17, 18, 19, 20, 27, 29, 30, 31, 32, 33, 40], and pseudo almost periodic functions [9, 11, 28]. Moreover, we recall that N'Guérékata has given a huge impulse to the study of almost automorphic solutions of differential equations [1, 7, 12, 14, 15, 21, 23, 25, 26]. For some recent results of almost automorphic differential equations consult also [5, 6].

In this paper, we are initially motivated by a biological-mathematical model [13, 16, 22, 37, 38, 41] which is a delayed differential equation of the following type

$$y'(t) = -\delta(t)y(t) + p(t)g(y(t-\tau)),$$

with  $\tau > 0$ ,  $\delta$  and  $p$  almost automorphic functions and  $g$  a Lipschitz function. Then, we focus our attention on the existence and uniqueness of solutions of the following delayed differential equation

$$y' = A(t)y + f(t) + g(t, y(t-\tau)) \quad \text{with } \tau \geq 0, \quad (1.1)$$

under several assumptions on  $A$ ,  $f$  and  $g$ . Naturally, the assumptions on  $A$  and  $f$  are related to the almost automorphic behavior and the assumptions on  $g$  are mainly related with a Lipschitz requirement. We note that (1.1) naturally, includes as particular cases the following equations

$$y' = A(t)y, \quad (1.2)$$

$$y' = A(t)y + f(t), \quad (1.3)$$

$$y' = A(t)y + f(t) + g(t, y(t)). \quad (1.4)$$

Thus, following a natural sequence of the classical systemic study of ordinary differential equations we start by analyzing the homogeneous linear equation (1.2). Then, we develop the theory for the non-homogeneous linear equation (1.3) by applying the method of variation of parameters. In a

---

*Date:* March 29, 2016.

*Key words and phrases.* Abstract delay differential equations, Almost automorphic, Exponential dichotomy, Ergodicity, Evolution operator.

Partially supported by FONDECYT 1120709. CONICYT-PCHA/Magíster Nacional/2013-221320155. Aníbal Coronel thanks for the support of research projects 124109 3/R, 104709 01 F/E and 121909 GI/C at Universidad del Bío-Bío, Chile.

third place, we analyze the nonlinear equation (1.4) by using the fixed point arguments. Finally, by a composition result of automorphic functions we get an extension of the results for (1.4) to the delay equation (1.1).

The main contributions and the organization of the paper are given as follows. In section 2 we introduce the general assumptions, recall the concepts of almost automorphicity and ergodic functions, define a convolution operator and get the results for (1.2). To be a little more precise in this section, we obtain some conditions for the exponential dichotomy using ergodic functions and we prove the Bi-almost periodicity and Bi-almost automorphicity of the Green function when the evolution operator commute with the projection. We note that, the integral Bi-almost automorphicity property of the Green function is fundamental to obtain the main results. In section 3, almost automorphicity of solutions of nonautonomous systems (1.3), (1.4) and (1.1) are obtained. Here, the results of almost automorphicity of the differential equation solutions are obtained by assuming that  $A$  and  $f$  are almost automorphic and  $g$  satisfies (2.1). Finally, in section 4 we study a biological model establishing an explicit condition under which there exist a unique almost automorphic solution of the Lasota-Ważewska equation.

## 2. PRELIMINARIES

In this section we present some general assumptions, precise the concepts related with the almost automorphic notion, we recall the notion of ergodic functions, we define a convolution operator and the  $\alpha$ -exponential dichotomy and introduce several results for the homogeneous equation (1.2) in the scalar, system and abstract case.

**2.1. General assumptions.** Here we present two general assumptions. Firstly, throughout of the paper  $(V, \|\cdot\|_V)$  will be a Banach space and let  $(BC(\mathbb{R}, V), \|\cdot\|_\infty)$  will be used to denote the Banach space of bounded continuous functions from  $\mathbb{R}$  into  $V$  endowed with the sup norm  $\|\varphi\|_\infty = \sup_{t \in \mathbb{R}} \|\varphi(t)\|_V$ . Second, concerning to the assumptions on the coefficients  $A, f$  and  $g$  for equations (1.1)-(1.4) we comment that it will be specifically done on the hypothesis of each result. However, in order to give a unified presentation, we introduce some notation related to the assumption of the local Lipschitz behavior of  $g$ . Indeed, given a function  $g$ , it is assumed that:

$$\left. \begin{aligned} (g_0) \quad & g(t, 0) = 0 \text{ for all } t \in \mathbb{R}; \\ (g_1) \quad & \text{The function } g(t, y) \text{ is continuous on } \mathbb{R} \times \Delta(\varphi_0, \rho) \text{ with } \Delta(\varphi_0, \rho) \\ & \text{the open ball centred in a given (fix) function } \varphi_0 : \mathbb{R} \rightarrow V \text{ and with} \\ & \text{radius } \rho \in \mathbb{R}^+, \text{ i.e., } \Delta(\varphi_0, \rho) = \left\{ \varphi : \mathbb{R} \rightarrow V \mid \|\varphi - \varphi_0\|_\infty \leq \rho \right\}. \text{ In par-} \\ & \text{ticular, in subsections 3.2-3.3 will be assumed that } \varphi_0 \text{ is of the form} \\ & \varphi_0(t) = \int_{\mathbb{R}} G(t, s) f(s) ds \text{ with } G \text{ the Green function defined on (3.17);} \\ (g_2) \quad & \text{There exist a positive constant } L \text{ such that the inequality} \\ & \|g(t, y_1) - g(t, y_2)\| \leq L \|y_1 - y_2\| \text{ holds for all } (t, y_1, y_2) \in \mathbb{R} \times \Delta(\varphi_0, \rho)^2. \end{aligned} \right\} \quad (2.1)$$

This set of conditions  $(g_0)$ -( $g_2$ ) appear in several parts of the paper and essentially when we study the nonlinear equations in subsections 3.2-3.3.

**2.2. Almost automorphic notion and related concepts.** We recall that the almost automorphic functions have been developed by Bochner [2, 3] as a generalization of almost periodic functions. We recall that a function  $f \in BC(\mathbb{R}, V)$  is called Bohr almost periodic [8] if for each  $\epsilon > 0$ , there exist  $l_\epsilon > 0$  such that every interval of length  $l_\epsilon$  contains a number  $\xi$  with the property:  $\|f(t + \xi) - f(t)\|_V \leq \epsilon$  for  $t \in \mathbb{R}$ . The set of Bohr almost periodic will be denoted by  $BC(\mathbb{R}, V)$ . Then, we precise the concept of almost automorphic functions and matrices.

**Definition 2.1.** Consider  $V$  a Banach space. Then,

- (i) A continuous function  $\psi : \mathbb{R} \rightarrow V$  is called an almost automorphic function if for any sequence of real numbers  $\{\tilde{\tau}_n\}_{n=1}^\infty$ , there exist a subsequence  $\{\tau_n\}_{n=1}^\infty$  of  $\{\tilde{\tau}_n\}_{n=1}^\infty$  such that the limit of the sequence  $\{\psi(t + \tilde{\tau}_n)\}_{n=1}^\infty$ , denoted by  $\tilde{\psi}(t)$ , is well defined for all  $t \in \mathbb{R}$  and the sequence  $\{\tilde{\psi}(t - \tilde{\tau}_n)\}_{n=1}^\infty$  converges pointwise on  $\mathbb{R}$  to  $\psi(t)$ , or equivalently

$$\tilde{\psi}(t) = \lim_{n \rightarrow \infty} \psi(t + \tau_n) \quad \text{and} \quad \psi(t) = \lim_{n \rightarrow \infty} \tilde{\psi}(t - \tau_n) \quad (2.2)$$

are well defined for all  $t \in \mathbb{R}$ . The collection of all almost automorphic functions from  $\mathbb{R}$  to  $V$  is denoted by  $AA(\mathbb{R}, V)$ .

- (ii) A matrix valued function  $A : \mathbb{R} \rightarrow \mathbb{C}^{d_1 \times d_2}$  is called an almost automorphic matrix valued function or equivalently (most of the time by briefness)  $A(t) \in \mathbb{C}^{d_1 \times d_2}$  is called an almost automorphic matrix if for any sequence  $\{\xi'_n\}_{n=1}^\infty \subset \mathbb{R}$ , there exist a subsequence  $\{\xi_n\}_{n=1}^\infty$  of  $\{\xi'_n\}_{n=1}^\infty$  and a matrix  $B(t) \in \mathbb{C}^{d_1 \times d_2}$  such that the sequences  $\{A(t + \xi_n)\}_{n=1}^\infty$  and  $\{B(t - \xi_n)\}_{n=1}^\infty$  converges pointwise to  $B(t)$  and  $A(t)$ , respectively.

We note that the convergence in (2.2) is pointwise. Then, the function  $\tilde{\psi}$  in (2.2) is measurable, but not necessarily continuous. Moreover, we note if we consider that convergence in definition 2.1 is uniform on  $\mathbb{R}$  instead of pointwise convergence, we get that the function  $\psi$  is Bochner almost periodic. It is well known that both definitions of almost periodicity (Bohr and Bochner) are equivalents, see for instance [8]. Now, we note that  $AP(\mathbb{R}, V)$  and  $AA(\mathbb{R}, V)$  are vectorial space and  $AP(\mathbb{R}, V)$  is a proper subspace of  $AA(\mathbb{R}, V)$ , since for instance  $\psi(t) = \cos([2 + \sin(t) + \sin(t\sqrt{2})]^{-1})$  is an almost periodic function but not almost automorphic. Similarly, it is proven that the inclusion  $AP(\mathbb{R}, V) \subset BC(\mathbb{R}, V)$ , for an extensive discussion consult [4, 3, 15, 14, 21, 23, 24, 25, 26, 34, 35, 42, 44, 10, 1, 39, 43, 12].

To close this subsection we introduce two additional facts. Firstly, we note that the simpler equation (1.3) with  $A \equiv 0$ , i.e.,  $y'(t) = f(t)$ , with  $f \in AA(\mathbb{R}, V)$  has not necessarily a solution  $y \in AA(\mathbb{R}, V)$ . However, this fact is true in a uniformly convex Banach space  $V$  and hence in every Hilbert space, see Theorem 2.1. In the second place we need a composition result [7], which will be fundamental for the analysis of (1.4) and (1.1), see Proposition 2.2.

**Theorem 2.1.** Denote by  $C_0$  the vectorial space formed by the functions which vanishes at infinity. Consider that  $V$  is a Banach space which does not contain  $C_0$  as an isomorphic subspace and let  $f \in AA(\mathbb{R}, V)$ . Then, the function  $F(t) = \int_0^t f(s)ds$  is in  $AA(\mathbb{R}, V)$  if and only if it is bounded. Such Banach space with this property for  $F$  will be called a Banach space with the Bohl-Bohr property.

**Proposition 2.2.** Let  $g = g(t, y) \in AA(\mathbb{R} \times V, V)$  uniformly in  $t$  for  $y$  in a compact set contained in  $V$  and  $g$  satisfies the assumptions given on (2.1). Then,  $g(t, \varphi(t)) \in AA(\mathbb{R}, V)$  for all  $\varphi \in AA(\mathbb{R}, \Delta(\varphi_0, \rho))$ .

**2.3. Ergodic functions.** Here we introduce the concept of ergodic functions and deduce that these types of functions implies naturally an exponential behavior (or  $\alpha$ -exponential dichotomy to be more precise).

**Definition 2.2.** A function  $f \in BC(\mathbb{R}, V)$  is called an ergodic function if the limit

$$M(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T+\xi}^{T+\xi} f(s)ds$$

exists uniformly with respect to  $\xi \in \mathbb{R}$  and its value is independent of  $\xi$ . The complex number  $M(f)$  is called the mean of the function  $f$ .

The mean of an ergodic function has several properties, a complete list of properties may be consulted in [45]. Among these useful basic properties, we only recall the translation invariance property, since it will be used frequently in the proofs given below in this paper. Indeed, the translation invariance property of  $M(f)$ , set that  $M(f)$  satisfies the following identity

$$M(f) = M(f_\xi), \quad (2.3)$$

where  $f_\xi$  denotes a  $\xi$ -translation of  $f$ , i.e.  $f_\xi(t) = f(t + \xi)$  for all  $t \in \mathbb{R}$  and any arbitrarily given  $\xi \in \mathbb{R}$  (but fixed).

**Lemma 2.3.** Consider  $\mu \in BC(\mathbb{R}, \mathbb{C})$  an ergodic function with  $\text{Re}(M(\mu)) \neq 0$  and also consider  $\alpha \in \mathbb{R}^+$ . Then, there exist two positive constants  $T_0$  (big enough) and  $c$  such that the two assertions given below are valid:

(i) If  $\operatorname{Re}(M(\mu)) \in ]-\infty, -\alpha[ \subset \mathbb{R}^-$ , then the following inequalities hold true:

$$\operatorname{Re} \left( \int_s^t \mu(r) dr \right) < -\alpha(t-s) \quad \text{for } t-s > T_0, \quad (2.4)$$

$$\left| \exp \left( \int_s^t \mu(r) dr \right) \right| \leq c \exp(-\alpha(t-s)) \quad \text{for all } (t, s) \in \mathbb{R}^2 \text{ such that } t \geq s \geq 0. \quad (2.5)$$

(ii) If  $\operatorname{Re}(M(\mu)) \in ]\alpha, \infty[ \subset \mathbb{R}^+$ , then the following inequalities hold true:

$$\operatorname{Re} \left( \int_s^t \mu(r) dr \right) < \alpha(t-s) \quad \text{for } s-t > T_0, \quad (2.6)$$

$$\left| \exp \left( \int_s^t \mu(r) dr \right) \right| \leq c \exp(\alpha(t-s)) \quad \text{for all } (t, s) \in \mathbb{R}^2 \text{ such that } s \geq t \geq 0. \quad (2.7)$$

*Proof.* Let us assume that  $\mu \in BC(\mathbb{R}, \mathbb{C})$  is an ergodic function. Then, by the Definition 2.2 and the translation invariance property of  $M(f)$  (see (2.3)) we have that

$$\int_0^T \mu(t+\tau) d\tau = [M(\mu) + o(1)]T \quad \text{when } T \rightarrow \infty. \quad (2.8)$$

Here and throughout of the paper  $o(1)$  corresponds to the well known Bachmann-Landau notation, i.e.  $f = o(g)$  if and only if  $(f/g)(x) \rightarrow 0$  when  $x \rightarrow \infty$ . Then, when  $T = t-s$ , we get

$$\int_s^t \mu(r) dr = \int_0^T \mu(-s+\tau) d\tau \quad (2.9)$$

and the proof of (2.4) follows immediately. The proof of (i) is a consequence of the exponential function increasing behavior. Thus, the proof of item (i) is completed. Now, the proof of item (ii) is similar and we omit it.  $\square$

**2.4. The convolution operator.** Let us denote by  $L^1(\mathbb{R})$  and  $L^\infty(\mathbb{R})$  the spaces of Lebesgue integrable functions on  $\mathbb{R}$  and essentially bounded functions on  $\mathbb{R}$ , respectively. Then, the convolution operator on  $\mathcal{L} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$  is defined as the operator such that

$$\mathcal{L}(\varphi)(t) = \int_{\mathbb{R}} h(t-s)\varphi(s)ds, \quad h \in L^1(\mathbb{R}), \quad t \in \mathbb{R}, \quad (2.10)$$

for all  $\varphi \in L^\infty(\mathbb{R})$ . Some properties of  $\mathcal{L}$ , which will be needed in the proof the main results, are summarized in the following lemma.

**Lemma 2.4.** *Consider  $\mathcal{L}$  the convolution operator defined by (2.10). Then, the spaces  $BC(\mathbb{R})$ ,  $AP(\mathbb{R})$  and  $AA(\mathbb{R})$  are invariants under the operator  $\mathcal{L}$ . Moreover, the inequalities*

$$\|\mathcal{L}\varphi\|_{L^\infty(\mathbb{R})} \leq \|\varphi\|_{L^\infty(\mathbb{R})}\|h\|_{L^1(\mathbb{R})} \quad \text{for } \varphi \in BC(\mathbb{R}), \quad (2.11)$$

$$\|(\mathcal{L}\varphi)_\xi - \mathcal{L}\varphi\|_{L^\infty(\mathbb{R})} \leq \|(\varphi)_\xi - \varphi\|_{L^\infty(\mathbb{R})}\|h\|_{L^1(\mathbb{R})} \quad \text{for } \xi \in \mathbb{R} \text{ and } \varphi \in AP(\mathbb{R}), \quad (2.12)$$

hold. Here  $(\mathcal{L}\varphi)_\xi$  and  $(\varphi)_\xi$  are the  $\xi$ -translation functions for  $\mathcal{L}\varphi$  and  $\varphi$ , respectively.

*Proof.* Let us select  $\varphi \in AA(\mathbb{R})$ . Then, by Definition 2.1, given an arbitrary sequence of real numbers  $\{\tilde{\tau}_n\}_{n=1}^\infty$ , there exist a subsequence  $\{\tau_n\}_{n=1}^\infty$  of  $\{\tilde{\tau}_n\}_{n=1}^\infty$  such that (2.2) is satisfied. Now, if we consider  $\psi = L(\varphi)$ , we have that (2.2) is equivalently rewritten as follows

$$\tilde{\psi}(t) = \lim_{n \rightarrow \infty} \psi(t + \tau_n) \quad \text{and} \quad \psi(t) = \lim_{n \rightarrow \infty} \tilde{\psi}(t - \tau_n). \quad (2.13)$$

Indeed, this fact can be proved by application of Lebesgue's dominated convergence theorem, since

$$\begin{aligned} \tilde{\psi}(t) &= \lim_{n \rightarrow \infty} \mathcal{L}(\varphi_{\tau_n})(t) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} h(r)\varphi_{\tau_n}(t-r)dr \\ &= \int_{\mathbb{R}} h(r)\tilde{\varphi}(t-r)dr = \mathcal{L}(\tilde{\varphi})(t). \end{aligned}$$

Let us consider  $\varphi \in BC(\mathbb{R})$ , then we deduce (2.11) by application of the Hölder inequality. Now, from (2.11) we follow the invariance of  $BC(\mathbb{R})$ .  $\square$

We note that, if we define

$$h_1(x) = \begin{cases} \exp(-\alpha x), & x > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad h_2(x) = \begin{cases} \exp(\alpha x), & x < 0, \\ 0, & \text{otherwise,} \end{cases}$$

for some  $\alpha \in \mathbb{R}^+$  and denote by  $\mathcal{L}_1$  and  $\mathcal{L}_2$  the corresponding convolution operators associated with  $h_1$  and  $h_2$ , respectively. Then, we get an interesting result by application of Lemma 2.4. More precisely, we have the following Corollary.

**Corollary 2.5.** *Consider  $\alpha \in \mathbb{R}^+$  and the operators  $\mathcal{L}_i$ ,  $i = 1, 2$  defined by*

$$\mathcal{L}_1(\varphi)(t) = \int_{-\infty}^t e^{-\alpha(t-s)} \varphi(s) ds \quad \text{and} \quad \mathcal{L}_2(\varphi)(t) = \int_t^{\infty} e^{\alpha(t-s)} \varphi(s) ds, \quad (2.14)$$

*respectively. Then, the spaces  $BC(\mathbb{R})$ ,  $AP(\mathbb{R})$  and  $AA(\mathbb{R})$  are invariants under the operator  $\mathcal{L}_i$ ,  $i = 1, 2$ . Moreover, the inequalities (2.11) and (2.12) are satisfied with  $\mathcal{L}_i$ ,  $i = 1, 2$ , instead of  $\mathcal{L}$ .*

**2.5. Some concepts and properties related to equation (1.2).** In this subsection we study the equation (1.2). In order to introduce the concepts and results, we recall the standard notation of fundamental matrix and flow associated with (1.2), which are denoted by  $\Phi_A$  and  $\Psi_A$ , respectively. More precisely

$$\left. \begin{array}{l} \text{Given a matrix } A(t), \text{ then the notation } \Phi_A = \Phi_A(t) \text{ and } \Psi_A \text{ are used for} \\ \text{a fundamental matrix of the system (1.2) and for the application defined} \\ \text{as follows } \Psi_A(t, s) = \Phi_A(t) \Phi_A^{-1}(s). \end{array} \right\} \quad (2.15)$$

**Lemma 2.6.** *Consider the notation (2.15). Then, the identities*

$$\Psi_A(t, s) - \Psi_B(t, s) = \int_s^t \Psi_B(t, r) [A(r) - B(r)] \Psi_A(r, s) dr, \quad (2.16)$$

$$\Psi_A(t + \xi, s + \xi) - \Psi_B(t, s) = \int_s^t \Psi_B(t, r) [A(r + \xi) - B(r)] \Psi_A(r + \xi, s + \xi) dr, \quad (2.17)$$

*are satisfied for all  $(t, s, \xi) \in \mathbb{R}^3$ .*

*Proof.* Let us denote by  $H$  the function defined by the following correspondence rule  $H(t, s) = \Psi_A(t, s) - \Psi_B(t, s)$ . Then, by partial differentiation of  $H$  with respect to the first variable and making some rearrangements, we get

$$\begin{aligned} \frac{\partial}{\partial t} H(t, s) &= \frac{\partial}{\partial t} \Psi_A(t, s) - \frac{\partial}{\partial t} \Psi_B(t, s) \\ &= (A(t) - B(t)) \Psi_A(t, s) + B(t) (\Psi_A(t, s) - \Psi_B(t, s)) \\ &= (A(t) - B(t)) \Psi_A(t, s) + B(t) H(t, s). \end{aligned}$$

Now, by multiplying to the left by  $\Psi_B(t, r)$  and simplifying, we deduce that

$$\begin{aligned} \Psi_B(t, r) (A(r) - B(r)) \Psi_A(r, s) &= \Psi_B(t, r) \frac{\partial}{\partial r} H(r, s) - \Psi_B(t, r) B(r) H(r, s) \\ &= \Psi_B(t, r) \frac{\partial}{\partial r} H(r, s) - \left( \frac{\partial}{\partial r} \Psi_B(t, r) \right) H(r, s) \\ &= \frac{\partial}{\partial r} (\Psi_B(t, r) H(r, s)). \end{aligned}$$

Thus, by integration over the interval  $[s, t]$ , we have that

$$\int_s^t \Psi_B(t, r) [A(r) - B(r)] \Psi_A(r, s) dr = \Psi_B(t, t) H(t, s) - \Psi_B(t, s) H(s, s),$$

which implies (2.16) by noticing that  $\Psi_B(t, t) = I$  and  $H(s, s) = 0$ . Now, the proof (2.17) follows by similar arguments or by direct application of (2.16).  $\square$

**Lemma 2.7.** Consider the notation (2.15) and the sets  $\overrightarrow{\mathbb{R}^2}$  and  $\overleftarrow{\mathbb{R}^2}$  defined as follows

$$\overrightarrow{\mathbb{R}^2} = \{(s, t) \in \mathbb{R}^2 : s > t\} \quad \text{and} \quad \overleftarrow{\mathbb{R}^2} = \{(s, t) \in \mathbb{R}^2 : s < t\},$$

respectively. Assume that the following three statements are true:  $A(t)$  is an almost periodic matrix (see definition 2.1),  $P$  is a constant projection matrix that commutes with  $\Phi_A$  and for some given positive constants  $c$  and  $\alpha$  the inequality

$$\|\Psi_A(t, s)P\| \leq c \exp(-\alpha|t - s|), \quad (2.18)$$

is satisfied for all  $(t, s) \in \overrightarrow{\mathbb{R}^2}$  (or for all  $(t, s) \in \overleftarrow{\mathbb{R}^2}$ ). Then, for all  $(t, s, \xi) \in \overrightarrow{\mathbb{R}^2} \times \mathbb{R}$  (respectively  $(t, s) \in \overleftarrow{\mathbb{R}^2} \times \mathbb{R}$ ), there exist two real constants  $c_1 > 0$  and  $\alpha' \in ]0, \alpha[$ , such that

$$\|\Psi_A(t + \xi, s + \xi)P - \Psi_A(t, s)P\| \leq c_1 \|A(\cdot + \xi) - A(\cdot)\|_\infty \exp(-\alpha'|t - s|). \quad (2.19)$$

In particular, if  $\xi$  is an  $\epsilon$ -almost period of  $A$  the inequality

$$\|\Psi_A(t + \xi, s + \xi)P - \Psi_A(t, s)P\| \leq c_1 \epsilon \exp(-\alpha'|t - s|), \quad (2.20)$$

is satisfied for all  $(t, s, \xi) \in \overrightarrow{\mathbb{R}^2} \times \mathbb{R}$  (respectively  $(t, s) \in \overleftarrow{\mathbb{R}^2} \times \mathbb{R}$ ) or equivalently  $\Psi_A(t, s)P$  is Bi-almost periodic.

*Proof.* Since the proofs with  $(t, s, \xi) \in \overrightarrow{\mathbb{R}^2} \times \mathbb{R}$  or with  $(t, s, \xi) \in \overleftarrow{\mathbb{R}^2} \times \mathbb{R}$  are analogous, we consider only one of the cases. Then, in order to fix ideas, let us consider  $(t, s, \xi) \in \overrightarrow{\mathbb{R}^2} \times \mathbb{R}$  and recall the notation  $\Delta_\xi$  defined by

$$\Delta_\xi F(\mathbf{x}) = F(\mathbf{x} + \xi) - F(\mathbf{x}) \text{ for any function } F. \quad (2.21)$$

In particular, for instance, we have that  $\Delta_\xi \Phi(t, s) = \Phi(t + \xi, s + \xi) - \Phi(t, s)$  and  $\Delta_\xi A(t) = A(t + \xi) - A(t)$ . From (2.16) and the hypothesis that  $P$  is a constant projection matrix, i.e.  $P^2 = P$ , which commutes with  $\Phi(t)$  for every  $t \in \mathbb{R}$ , we have that

$$\Delta_\xi \Psi_A(t, s)P = \int_s^t \Psi_A(t, r)P \Delta_\xi A(r) \Psi_{A_\xi}(r, s)P dr. \quad (2.22)$$

Now, by the assumption (2.18) we follow that (2.22) implies the following estimate

$$\begin{aligned} \|\Delta_\xi \Psi_A(t, s)P\| &\leq c \exp(-\alpha|t - s|) \int_s^t \|\Delta_\xi A(r)\| dr \\ &\leq c \exp(-\alpha|t - s|) \|\Delta_\xi A\|_\infty |t - s|, \end{aligned}$$

which implies (2.19). The inequality (2.20) follows immediately from (2.19) using the fact that  $\xi$  is an  $\epsilon$ -almost period of  $A$ .  $\square$

**Definition 2.3.** Consider the notation (2.15). If the fact that  $A(t)$  is an almost automorphic matrix (see definition 2.1-(ii)) implies the following convergence

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t \left\| (\Psi_{A_{\xi_n}} - \Psi_B)(t, s)P \right\| ds = \lim_{n \rightarrow \infty} \int_{-\infty}^t \left\| (\Psi_{B_{-\xi_n}} - \Psi_A)(t, s)P \right\| ds = 0, \quad (2.23)$$

the application  $\Psi_A$  is called integrally Bi-almost automorphic on  $] -\infty, t]$ . Similarly, if the almost automorphic behavior of  $A(t)$  implies the following convergence

$$\lim_{n \rightarrow \infty} \int_t^\infty \left\| (\Psi_{A_{\xi_n}} - \Psi_B)(t, s)P \right\| ds = \lim_{n \rightarrow \infty} \int_t^\infty \left\| (\Psi_{B_{-\xi_n}} - \Psi_A)(t, s)P \right\| ds = 0, \quad (2.24)$$

the application  $\Psi_A$  is called integrally Bi-almost automorphic on  $[t, \infty[$ . Here  $\{\xi_n\}_{n=1}^\infty$  and  $B(t)$  denotes the subsequence and the matrix with the properties given on definition 2.1-(ii).

**Lemma 2.8.** Consider the notation (2.15). If the assumptions of Lemma 2.7 hold, then  $\Psi_A$  is integrally Bi-almost automorphic on  $] -\infty, t]$  and on  $[t, \infty[$ .



*Proof.* Let us consider that  $A(t)$  is an almost automorphic matrix. Then by definition 2.1-(ii) we follow that for any sequence  $\{\xi'_n\}_{n=1}^\infty \subset \mathbb{R}$ , there exist a subsequence  $\{\xi_n\}_{n=1}^\infty$  of  $\{\xi'_n\}_{n=1}^\infty$  and a matrix  $B(t)$  such that

$$\lim_{n \rightarrow \infty} A_{\xi_n}(t) = B(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} B_{-\xi_n}(t) = A(t) \quad \text{for all } t \in \mathbb{R}. \quad (2.25)$$

Now, from the identity (2.16), we deduce that the assumption (2.18) implies the following bounds

$$\begin{aligned} \left\| \left( \Psi_{A_{\xi_n}} - \Psi_B \right)(t, s)P \right\| &\leq c e^{-\alpha|t-s|} \left\| \int_s^t (A_{\xi_n} - B)(r)dr \right\| \\ &\leq c_1 e^{-\alpha'|t-s|} (\|A\| + \|B\|), \end{aligned} \quad (2.26)$$

$$\begin{aligned} \left\| \left( \Psi_{B_{-\xi_n}} - \Psi_A \right)(t, s)P \right\| &\leq c e^{-\alpha|t-s|} \left\| \int_s^t (B_{-\xi_n} - A)(r)dr \right\| \\ &\leq c_1 e^{-\alpha'|t-s|} (\|A\| + \|B\|), \end{aligned} \quad (2.27)$$

for all  $(t, s, n) \in \mathbb{R}^2 \times \mathbb{N}$  and some real constants  $c_1 > 0$  and  $\alpha' \in ]0, \alpha[$ . Then, by applying four times the Lebesgue's dominated convergence theorem we deduce the integrally Bi-almost automorphic property of  $\Psi_A$ . Indeed, firstly by the first limit given in (2.25) we deduce that for each  $(s, t) \in \mathbb{R}^2$  the integral  $\int_s^t \|(A_{\xi_n} - B)(r)\|dr$  converges to 0 when  $n \rightarrow \infty$ . Then, with this convergence in mind, in a second application, from (2.26) we get that for each  $t \in \mathbb{R}$  the integral  $\int_{-\infty}^t \|(\Psi_{A_{\xi_n}} - \Psi_B)(t, s)P\|ds$  converges to 0 when  $n \rightarrow \infty$ . Similarly, applying twice more the Lebesgue's theorem (in the second integral (2.25) and in the inequality (2.27)) we deduce that for each  $(s, t) \in \mathbb{R}^2$  the integral  $\int_s^t \|(B_{-\xi_n} - A)(r)\|dr$  converges to 0 when  $n \rightarrow \infty$  and for each  $t \in \mathbb{R}$  the integral  $\int_{-\infty}^t \|(\Psi_{B_{-\xi_n}} - \Psi_A)(t, s)P\|ds$  converges to 0 when  $n \rightarrow \infty$ . Thus, (2.23) holds and  $\Psi_A$  is integrally Bi-almost automorphic. The proof of (2.24) can be obtained by similar arguments.  $\square$

**Definition 2.4.** Consider the notation (2.15). The linear system (1.2) has an  $\alpha$ -exponential dichotomy if there exist a projection  $P$  and two positive constants  $c$  and  $\alpha$  such that for all  $(t, s) \in \mathbb{R}^2$  the estimate

$$\|G_A(t, s)\| \leq c \exp(-\alpha|t-s|) \quad \text{with} \quad G_A(t, s) = \begin{cases} \Phi_A(t)P\Phi_A^{-1}(s), & t \geq s, \\ -\Phi_A(t)(I-P)\Phi_A^{-1}(s), & \text{otherwise,} \end{cases} \quad (2.28)$$

is satisfied. The matrix  $G_A$  is called the Green matrix associated with the dichotomy.

**Lemma 2.9.** Consider the notation (2.15) and define the Green operator  $\Gamma$  as follows

$$(\Gamma\varphi)(t) = \int_{-\infty}^{\infty} G(t, s)\varphi(s)ds, \quad t \in \mathbb{R}.$$

Assume that  $A(t)$  is an almost automorphic matrix and (1.2) has an exponentially dichotomy such the its projection commutes with the fundamental matrix  $\Phi_A$ . Then, the following assertions are satisfied:

- (i) The Green matrix  $G_A$  is integrally Bi-almost automorphic.
- (ii) The spaces  $BC(\mathbb{R}, V)$ ,  $AP(\mathbb{R}, V)$  and  $AA(\mathbb{R}, V)$  are invariants under the operator  $\Gamma$ . Moreover, there exist two positive constants  $c_1$  and  $c_2$  such that the following inequalities

$$\begin{aligned} \|\Gamma\varphi\|_\infty &\leq \frac{\|\varphi\|_\infty}{\alpha} \quad \text{for } \varphi \in BC(\mathbb{R}, V), \\ \|(\Delta_\xi \Gamma\varphi)(t)\| &\leq c_1 \|\varphi\|_\infty \sum_{i=1}^2 \mathcal{L}_i(|\Delta_\xi A|) + c_2 \sum_{i=1}^2 \mathcal{L}_i(|\Delta_\xi \varphi|) \quad \text{for } \varphi \in AP(\mathbb{R}, V), \end{aligned}$$

are satisfied. Here  $\mathcal{L}_i$  and  $\Delta_\xi$  denotes the operators defined on (2.14) and (2.21), respectively.

*Proof.* The proofs of (i) and (ii) are straightforward. Indeed, for (i), let us consider  $A(t), B(t)$  and  $\{\xi_n\}_{n=1}^\infty$  as given in the proof of Lemma 2.8. Now, by the assumptions we can deduce that

$$\int_{-\infty}^{\infty} \|(G_{A_{\xi_n}} - G_B)(t, s)\| ds \rightarrow 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \|(G_{B_{-\xi_n}} - G_A)(t, s)\| ds \rightarrow 0 \quad \text{when } n \rightarrow \infty$$

where  $G_A$  is the Green matrix defined on (2.28). Thus, we can follow that  $G_A$  is integrally Bi-almost automorphic. Meanwhile, we follow the proof of (ii) by application of Lemmas 2.3, 2.4 and 2.9-(i).  $\square$

### 3. MAIN RESULTS

In this section we present several results of Massera type for (1.3) and (1.1) and related with the almost automorphic behavior of the  $A$  and  $f$ .

**3.1. Results for (1.3).** Here we present a result for the scalar abstract case, see Theorem 3.1. Then, we extend this result can to linear triangular systems and general linear constant systems, see Theorem 3.2. We also, present simple and useful relation between finite and infinite dimension is deduced from Theorem 3.3. Finally, we present to results on the general case, see Theorems 3.4 and 3.5.

**Theorem 3.1.** *Consider the equation (1.3) with  $A = \mu : \mathbb{R} \rightarrow \mathbb{C}$  and denote by  $g$  the application defined by*

$$g(s, t) = \exp \left( \int_s^t \mu(r) dr \right). \quad (3.1)$$

*Then, the following assertions are valid:*

- (i) *Assume that  $\mu$  is a function belongs to  $AA(\mathbb{R}, \mathbb{C})$  satisfying  $M(\text{Re}(\mu)) \neq 0$ . Also, assume that  $f$  is belongs to  $AA(\mathbb{R}, V)$ . Then, a solution  $y$  of equation (1.3) is bounded if and only if  $y \in AA(\mathbb{R}, V)$  or equivalently the unique solution of equation (1.3) belongs  $AA(\mathbb{R}, V)$  is given by*

$$y(t) = \begin{cases} \int_{-\infty}^t g(s, t) f(s) ds, & M(\text{Re}(\mu)) < 0, \\ -\int_t^{\infty} g(s, t) f(s) ds, & M(\text{Re}(\mu)) > 0. \end{cases} \quad (3.2)$$

- (ii) *Assume that  $\mu(t) = ia(t)$  with  $\int_0^t a(s) ds$  bounded and  $V$  a Bohl-Bohr Banach space. Then, a solution  $y$  of equation (1.3) is bounded if and only if  $y$  is belongs  $AA(\mathbb{R}, V)$  and is given by*

$$y(t) = \exp \left( i \int_0^t a(r) dr \right) v + \int_0^t \exp \left( i \int_s^t a(r) dr \right) f(s) ds \quad \text{for all } v \in V. \quad (3.3)$$

*Proof.* (i) Before of prove the item we deduce two estimates (see (3.5) and (3.6)) and introduce some notation (see (3.7) to (3.8)). Firstly, by Lemma 2.3 we can deduce that the scalar equation  $x' = \mu(t)x$  has an  $\alpha$ -exponential dichotomy. Indeed, we note that by the hypothesis  $M(\text{Re}(\mu)) \neq 0$  we can always select  $\alpha$  satisfying  $|M(\text{Re}(\mu))| \geq \alpha > 0$ . Then, by application of Lemma 2.3, we have that there exist a positive constant  $c$  such that

$$|g(t, s)| \leq ce^{-\alpha|t-s|} \quad \text{with } g \text{ defined in (3.1).} \quad (3.4)$$

Moreover, by application of Lemma 2.7 and integration on  $s$ , we deduce that there exist  $c_1 \in \mathbb{R}^+$  and  $\alpha' \in ]0, \alpha[$  such the following inequalities

$$\int_{-\infty}^t |\Delta_\xi g(t, s)| |f_\xi(s)| ds \leq \frac{c_1}{\alpha'} \|f\|_\infty \|\Delta_\xi \mu\|_\infty \quad \text{for all } t \in \mathbb{R}, \quad (3.5)$$

$$\int_t^{\infty} |\Delta_\xi g(t, s)| |f_\xi(s)| ds \leq \frac{c_1}{\alpha'} \|f\|_\infty \|\Delta_\xi \mu\|_\infty \quad \text{for all } t \in \mathbb{R}, \quad (3.6)$$



hold. Now, let us consider  $\{\xi_n\}_{n=1}^\infty$  a sequence in  $\mathbb{R}$ . Then, by the assumption  $\mu$  belongs  $AA(\mathbb{R}, V)$ , there exist a subsequence  $\{\xi'_n\}_{n=1}^\infty$  of  $\{\xi_n\}_{n=1}^\infty$  and the function  $\tilde{\mu}$  such that

$$\lim_{n \rightarrow \infty} \mu_{\xi'_n}(t) = \tilde{\mu}(t), \quad \lim_{n \rightarrow \infty} \tilde{\mu}_{-\xi'_n}(t) = \mu(t), \quad \text{for all } t \in \mathbb{R}. \quad (3.7)$$

Similarly, given the sequence  $\{\xi'_n\}_{n=1}^\infty$  by the hypothesis  $f \in AA(\mathbb{R}, V)$ , there exist a subsequence  $\{\xi''_n\}_{n=1}^\infty$  of  $\{\xi'_n\}_{n=1}^\infty$  and the function  $\tilde{f}$  such that

$$\lim_{n \rightarrow \infty} f_{\xi''_n}(t) = \tilde{f}(t), \quad \lim_{n \rightarrow \infty} \tilde{f}_{-\xi''_n}(t) = f(t), \quad \text{for all } t \in \mathbb{R}. \quad (3.8)$$

Now we develop the proof of the item. Indeed, we consider that  $y$  is defined by (3.2) and we prove that  $y$  is belongs to  $AA(\mathbb{R}, V)$ . Let us start by considering the notation  $y_\pm$  and  $\tilde{y}_\pm$  for the functions defined as follows

$$y_+(t) = \int_{-\infty}^t g(t, s) f(s) ds, \quad \tilde{y}_+(t) = \int_{-\infty}^t g(t, s) \tilde{f}(s) ds, \quad (3.9)$$

$$y_-(t) = - \int_t^\infty g(t, s) f(s) ds \quad \text{and} \quad \tilde{y}_-(t) = - \int_t^\infty g(t, s) \tilde{f}(s) ds, \quad (3.10)$$

respectively. Then, by algebraic rearrangements we deduce that

$$(y_+)_{\xi''_n}(t) - \tilde{y}_+(t) = \int_{-\infty}^t \Delta_{\xi''_n} g(t, s) f_{\xi''_n}(s) ds + \int_{-\infty}^t g(t, s) (f_{\xi''_n} - \tilde{f})(s) ds, \quad (3.11)$$

$$(y_-)_{\xi''_n}(t) - \tilde{y}_-(t) = - \int_t^\infty \Delta_{\xi''_n} g(t, s) f_{\xi''_n}(s) ds - \int_t^\infty g(t, s) (f_{\xi''_n} - \tilde{f})(s) ds, \quad (3.12)$$

Now, by using Lebesgue dominated convergence theorem we get that the four integrals on (3.11)-(3.12) converges to 0 when  $n \rightarrow \infty$ . Indeed, by (3.5) and (3.7) we follow that the first integral in (3.11) converges to 0 when  $n \rightarrow \infty$ . We see that the second integral in (3.11) vanishes when  $n \rightarrow \infty$  by consequence of (3.8). Meanwhile, we note that both integrals in (3.12) converge to 0 when  $n \rightarrow \infty$  by application of (3.6), (3.7) and (3.8). Consequently, we have that

$$\lim_{n \rightarrow \infty} (y_\pm)_{\xi''_n}(t) = \tilde{y}_\pm(t) \quad \text{for all } t \in \mathbb{R}. \quad (3.13)$$

Similarly, we can prove that  $(\tilde{y}_\pm)_{-\xi''_n}(t) \rightarrow y(t)$  for all  $t \in \mathbb{R}$  and when  $n \rightarrow \infty$ . Hence  $y \in AA(\mathbb{R}, V)$ .

(ii) Noticing that  $h(s) = \exp(i \int_0^s a) f(s) \in AA(\mathbb{R}, V)$  and the Banach space  $V$  has the Bohl-Bohr property we follow the proof by application of Theorem 2.1.  $\square$

**Theorem 3.2.** *Consider that  $A(t) \in AA(\mathbb{R}, \mathbb{C}^{p \times p})$  is an upper triangular of order  $p \times p$ . Then the following assertions are valid:*

- (i) *Assume that  $A(t)$  satisfies the condition  $\text{Re}(M(a_{ii})) \neq 0$  for all  $i = 1, \dots, p$  and  $f \in AA(\mathbb{R}, V^p)$ . Then, a solution  $y$  of (1.3) is bounded if and only if  $y \in AA(\mathbb{R}, V^p)$*
- (ii) *Assume that  $A(t)$  satisfies the condition*

$$a_{kk}(t) = i\beta_k(t) \quad \text{with} \quad \int_0^t \beta_k(s) ds \quad \text{bounded for all } k = 1, \dots, p. \quad (3.14)$$

*Assume that the Banach space  $V$  has the Bohl-Bohr property. Then any solution  $y$  of system (1.3) is bounded if and only if  $y$  is belongs  $AA(\mathbb{R}, V^p)$ .*

*Proof.* (i) If we consider that  $A(t)$  is triangular matrix, we have that the system (1.3) is of the following type

$$\begin{aligned} y'_1 &= a_{11}(t)y_1 + a_{12}(t)y_2 + a_{13}(t)y_3 + \dots + a_{1p}(t)y_p + f_1(t) \\ y'_2 &= a_{22}(t)y_2 + a_{23}(t)y_3 + \dots + a_{2p}(t)y_p + f_2(t) \\ &\vdots \\ y'_p &= a_{pp}(t)y_p + f_p(t). \end{aligned} \quad (3.15)$$

We note that the  $p$ -th equation in (3.15) can be analyzed by application of Theorem 3.1. Indeed, by Theorem 3.1-(i), we have that there exist  $y_p \in AA(\mathbb{R}, V)$  given for

$$y(t) = \begin{cases} \int_{-\infty}^t g_p(s, t) f(s) ds, & M(\operatorname{Re}(\mu_p)) < 0, \\ -\int_t^{\infty} g_p(s, t) f(s) ds, & M(\operatorname{Re}(\mu_p)) > 0. \end{cases} \quad \text{with } g_p(s, t) = \exp\left(\int_s^t a_{pp}(r) dr\right).$$

Similarly, by substituting  $y_p \in AA(\mathbb{R}, V)$  in  $(p-1)$ -th equation of (3.15) and by a new application of Theorem 3.1-(i) we can find an explicit expression for  $y_{p-1}(t)$ . This argument can be repeated to construct  $y_{p-2}(t), y_{p-3}(t), \dots, y_2(t)$  and  $y_1(t)$  by backwards substitution and application of Theorem 3.1-(i) in the system (3.15). Hence, we can construct  $y(t)$  and also get that the conclusion of the theorem 3.2-(i) is valid.

(ii) The proof of this item is similar to the proof of the precedent item (i) of the Theorem 3.2. In a broad sense, in this case we apply Theorem 3.1-(ii) instead of Theorem 3.1-(i) and similarly we use backwards substitution.  $\square$

**Theorem 3.3.** *Let  $V$  be a Banach space having the Bohl-Bohr property. Let  $\{\mu_i\}_{i=1}^p$  be the eigenvalues of the  $p \times p$  constant matrix  $A$  satisfying  $|\mu_i| = 1$ . Then any bounded solution of (1.3)  $y \in AA(\mathbb{R}, V^p)$ . When all the eigenvalues  $\mu_i$  are distinct, these solutions have the form*

$$y(t) = \exp(At) \left[ v + \int_0^t \exp(-As) f(s) ds \right] \quad \text{for all } v \in V^p. \quad (3.16)$$

In the general case, a formula for the bounded solutions can be also obtained.

*Proof.* If  $\{\mu_i\}_{i=1}^p$  are distinct, the constant system (1.2) is similar to a diagonal system. Then, without loss of generality, we can suppose that  $A$  is an upper triangular matrix. Hence, the result (3.16) follows by application of the Theorem 3.2-(i).  $\square$

**Theorem 3.4.** *Consider  $A : V \rightarrow V$  an infinitesimal generator of a  $C_0$  group of bounded linear operators  $T(t)$  with  $t \in \mathbb{R}$  and define the Green function*

$$G(t, s) = \begin{cases} T(t-s)P, & t \geq s \\ -T(t-s)(I-P), & t \leq s. \end{cases} \quad (3.17)$$

Assume that  $A$  has an  $\alpha$ -exponential dichotomy, i.e., there exist two constants  $c$  and  $\alpha$  such that

$$\|G(t, s)\| \leq ce^{-\alpha|t-s|} \quad \text{for all } (t, s) \in \mathbb{R}^2. \quad (3.18)$$

Then if  $f \in AA(\mathbb{R}, V)$ , equation (1.3) has a unique solution  $y \in AA(\mathbb{R}, V)$  given by

$$y(t) = \int_{\mathbb{R}} G(t, s) f(s) ds \quad \text{for all } t \in \mathbb{R} \quad (3.19)$$

and satisfying the following estimate

$$\|y\|_{\infty} \leq \frac{2c}{\alpha} \|f\|_{\infty}. \quad (3.20)$$

*Proof.* From (3.17) and (3.18) and  $y \in BC(\mathbb{R}, V)$

$$\lim_{s \rightarrow \pm\infty} \|G(t, s)y(s)\| = 0 \quad (3.21)$$

Indeed, for  $t \geq s$  we have that  $\|G(t, s)y(s)\| \leq ce^{-\alpha(t-s)}\|y\|_{\infty}$ , which implies that  $\|G(t, s)y(s)\| \rightarrow 0$  when  $s \rightarrow -\infty$ . Similarly, we get that  $\|G(t, s)y(s)\|$  vanishes when  $s \rightarrow \infty$ . We note that, for  $t \in \mathbb{R}$  (fix), applying  $T(t-s)$  on the identity  $y'(s) = Ay(s) + f(s)$  and using the fact that  $A$  commutes with  $T(t)$  on the domain of  $A$ , we get

$$\begin{aligned} T(t-s)y'(s) &= T(t-s)Ay(s) + T(t-s)f(s) \\ &= AT(t-s)y(s) + T(t-s)f(s). \end{aligned} \quad (3.22)$$

Now, the proof consists of three main parts: (a) we prove that the solution  $y$  of (1.3) is given by (3.19); (b) we prove that  $y \in AA(\mathbb{R}, V)$  and (c) we prove the uniqueness.

(a). *Proof of that the solution  $y$  of (1.3) is given by (3.19).* Firstly, we note that a formal integration of (3.22) on  $s \in (-\infty, t)$  gives the following identity

$$\int_{-\infty}^t T(t-s)Py'(s)ds = \int_{-\infty}^t AT(t-s)Py(s)ds + \int_{-\infty}^t T(t-s)Pf(s)ds. \quad (3.23)$$

Moreover we have that

$$\frac{d}{ds}T(t-s)y(s) = -AT(t-s)y(s) + T(t-s)y'(s). \quad (3.24)$$

Then an integration on  $s \in [r, t]$  implies the following relation

$$Py(t) - T(t-r)Py(r) = - \int_r^t AT(t-s)Py(s)ds + \int_r^t T(t-s)Py'(s)ds. \quad (3.25)$$

Now, by (3.21) letting  $r \rightarrow -\infty$ , we deduce that

$$Py(t) = - \int_{-\infty}^t AT(t-s)Py(s)ds + \int_{-\infty}^t T(t-s)Py'(s)ds. \quad (3.26)$$

Here, we note that a integration of (3.22) on  $s \in [t, \infty)$  with  $Q = I - P$  gives

$$\int_t^\infty T(t-s)Qy'(s)ds = \int_t^\infty AT(t-s)Qy(s)ds + \int_t^\infty T(t-s)Qf(s)ds. \quad (3.27)$$

and a integration of (3.24) on  $s \in [t, r]$  yields

$$T(t-r)Qy(r) - Qy(t) = \int_t^r -AT(t-s)Qy(s)ds + \int_t^r T(t-s)Qy'(s)ds.$$

Now, by (3.21) and letting  $r \rightarrow \infty$  in the last relation we get

$$-(I-P)y(t) = - \int_t^\infty AT(t-s)(I-P)y(s)ds + \int_t^\infty T(t-s)(I-P)y'(s)ds. \quad (3.28)$$

The relation (3.28) together with (3.27) yields

$$-(I-P)y(t) = \int_t^\infty T(t-s)(I-P)f(s)ds. \quad (3.29)$$

Then, from (3.23), (3.26) and (3.29) we obtain

$$\begin{aligned} y(t) &= Py(t) + (I-P)y(t) \\ &= \int_{-\infty}^t T(t-s)Pf(s)ds - \int_t^\infty T(t-s)(I-P)f(s)ds \\ &= \int_{\mathbb{R}} G(t,s)f(s)ds. \end{aligned} \quad (3.30)$$

Thus, we conclude the proof of (3.19).

(b). *Proof  $y \in AA(\mathbb{R}, V)$ .* The proof of this property follows by (3.30) the hypothesis  $f \in AA(\mathbb{R}, V)$  and application of Lemma 2.9.

(c). *Proof of uniqueness of bounded solutions.* The uniqueness of the bounded solution for (1.3) follows by the fact that  $x \equiv 0$  is the unique bounded solution on  $\mathbb{R}$  of the linear equation (1.2). Indeed, if  $x \in BC(\mathbb{R}, V)$  is a solution of the linear system we have that  $x(t) = Px(t) + (I-P)x(t) = x_1(t) + x_2(t)$ . Note that  $x_1 \rightarrow \infty$  as  $t \rightarrow -\infty$  and  $x_2 \rightarrow -\infty$  as  $t \rightarrow \infty$ , by the exponential dichotomy.

Finally, we note that (3.20) is a consequence of (3.30).  $\square$

**Theorem 3.5.** Assume that  $A \in AA(\mathbb{R}, \mathbb{C}^{p \times p})$  and (1.2) has an exponential dichotomy with a projection  $P$  that commutes with the fundamental matrix  $\Phi_A(t)$ . Assume that  $f \in AA(\mathbb{R}, V^p)$ . Then, the linear non-homogeneous equation (1.3) has a unique  $AA(\mathbb{R}, V^p)$  solution given by

$$y(t) = \int_{\mathbb{R}} G(t, s) f(s) ds,$$

satisfying (3.20).

*Proof.* By application of Lemma 2.9.  $\square$

**Theorem 3.6.** Consider  $V$  be a Hilbert space and  $A$  a linear compact operator on  $V$ . Suppose that  $V = \oplus_{k=1}^{\infty} V_k$  is a Hilbert sum such that  $V_k$  is a finite dimensional subspace of  $V$  for each  $k \in \mathbb{N}$ . Suppose that each orthogonal projection  $P_k$  on  $V_k$  commutes with  $A$ . If  $f \in AA(\mathbb{R}, V)$ , then every bounded solution  $y$  of (1.3) is belongs  $AA(\mathbb{R}, V)$ .

*Proof.* Noticing that for any  $y \in V$ , we have that  $y = \sum_{k=1}^{\infty} P_k y = \sum_{k=1}^{\infty} y_k$ . Then, by the fact that  $A$  is bounded on  $V$ , we deduce that

$$Ay = \sum_{k=1}^{\infty} Ay_k = \sum_{k=1}^{\infty} AP_k y = \sum_{k=1}^{\infty} P_k Ay. \quad (3.31)$$

Now, from the hypothesis that  $f \in AA(\mathbb{R}, V)$ , we have that for any subsequence  $\{\tilde{\tau}_n\}_{n=1}^{\infty} \subset \mathbb{R}$ , there exist a subsequence  $\{\tau_n\}_{n=1}^{\infty} \subset \{\tilde{\tau}_n\}_{n=1}^{\infty}$  and a function  $\tilde{f}$  such that  $f_{\tau_n}(t) \rightarrow \tilde{f}(t)$  and  $\tilde{f}_{-\tau_n}(t) \rightarrow f(t)$  pointwise on  $\mathbb{R}$  when  $n \rightarrow \infty$ . Then, by compactness of  $A$  we deduce that  $Af_{\tau_n}(t) \rightarrow A\tilde{f}(t)$  and  $A\tilde{f}_{-\tau_n}(t) \rightarrow Af(t)$  pointwise on  $\mathbb{R}$  when  $n \rightarrow \infty$ . Now, choosing  $y_k(t) = P_k y(t)$  and assuming that  $y$  is solution of equation (1.3) we can deduce that

$$\begin{aligned} y'_k = P_k y' &= P_k (Ay(t) + f(t)) \\ &= AP_k y(t) + P_k f(t) \\ &= Ay_k(t) + P_k f(t) \end{aligned}$$

or equivalently  $y_k$  satisfies the equation (1.3) in the finite dimensional space  $V_k$  with  $P_k f(t) \in AA(\mathbb{R}, V_k)$  since  $P_k$  is a bounded linear operator. Thus,  $y_k$  is bounded if and only if  $y_k \in AA(\mathbb{R}, V_k)$ . Now, if  $y(t)$  is bounded the set  $\{Ay(t) | t \in \mathbb{R}\}$  is relatively compact in  $V$ . Hence  $\sum_{k=1}^{\infty} P_k Ay(t) = Ay(t)$  uniformly on  $\mathbb{R}$ .

On the other hand  $P_k Ay(t) \in AA(\mathbb{R}, V_k)$  since  $P_k Ay(t) = AP_k y(t) = Ay_k(t)$ . Then,  $Ay(t) \in AA(\mathbb{R}, V)$  and  $y'(t) \in AA(\mathbb{R}, V)$  since  $y(t)$  satisfies the equation (1.3). Therefore, using the Theorem 2.1,  $y \in AA(\mathbb{R}, V)$  since  $y \in BC(\mathbb{R}, V)$  and  $V$  is a Hilbert space.  $\square$

**3.2. Results for (1.4).** Before start we recall the notation  $\Delta(\varphi_0, \rho)$  and  $\varphi_0$  given on (2.1). Here, in this subsection, we present two results for (1.4) assuming fundamentally that  $g$  satisfies the assumptions given on (2.1) and  $f$  if a function such that the inequality

$$\|f\| \leq \frac{\alpha \rho}{2c}, \quad (3.32)$$

is satisfied for some positive constants  $\alpha$  and  $c$ , then  $0 \in \Delta(\varphi_0, \rho)$  or equivalently  $\|\varphi_0\| \leq \rho$ .

**Theorem 3.7.** Consider  $A(t) \in AA(\mathbb{R}, \mathbb{C}^{p \times p})$  such that (1.2) has an  $\alpha$ -exponential dichotomy with a projection  $P$  that commutes with  $\Phi_A$ . Assume that  $g$  satisfies the assumptions given on (2.1) and  $f$  is selected such that the (3.32) holds. Then, if  $4cL < \alpha$  the equation (1.4) has a unique solution belongs  $AA(\mathbb{R}, V^p)$ .

*Proof.* Let  $\Delta = AA(\mathbb{R}, V^p) \cap \Delta(\varphi_0, \rho)$  and  $G = G(t, s)$  the Green matrix associated to  $\alpha$ -exponential dichotomy, i.e

$$\|G(t, s)\| \leq ce^{-\alpha|t-s|} \quad \text{for some } c, \alpha \in \mathbb{R}^+ \text{ and for all } (t, s) \in \mathbb{R}^2.$$

Now, for  $\varphi \in \Delta$ , by Proposition 2.2 we follow that the function  $g(t, \varphi(t)) \in AA(\mathbb{R}, V^p)$ . Then, by Theorem 3.5, we have that

$$(\Gamma\varphi)(t) = \int_{\mathbb{R}} G(t, s)[f(s) + g(s, \varphi(s))]ds \quad (3.33)$$

is the unique  $AA(\mathbb{R}, V^p)$  solution of the system (1.4). Moreover,  $(\Gamma\varphi)(t)$  satisfies the inequality

$$\|\Gamma\varphi - \varphi_0\| \leq \frac{2cL}{\alpha}\|\varphi\| \leq \frac{2cL}{\alpha}(\|\varphi - \varphi_0\| + \|\varphi_0\|) \leq \frac{4cL}{\alpha}\rho \leq \rho.$$

Thus,  $\Delta$  is invariant under  $\Gamma$ . Furthermore, we note that  $\Gamma$  is a contraction, since

$$\begin{aligned} |(\Gamma\varphi_1)(t) - (\Gamma\varphi_2)(t)| &\leq L \int_{\mathbb{R}} |G(t, s)| |\varphi_1(s) - \varphi_2(s)| ds \\ &\leq \frac{2cL}{\alpha} \|\varphi_1 - \varphi_2\|. \end{aligned}$$

Hence by the Banach fixed point arguments, we deduce that  $\Gamma$  has a unique fixed point  $\varphi \in \Delta$ . Consequently, the function  $\varphi$  is the unique solution of (1.4).  $\square$

**Theorem 3.8.** *Consider that  $A$  is a linear operator satisfying the assumptions given on Theorem 3.4. Assume that  $g$  satisfies the assumptions given on (2.1) and  $f$  is selected such that the (3.32) holds. Then, if  $4cL < \alpha$  the equation (1.4) has a unique mild solution belongs  $AA(\mathbb{R}, V^p)$ .*

*Proof.* Let  $\Delta = AA(\mathbb{R}, V) \cap \Delta(\varphi_0, \rho)$  and  $\varphi \in AA(\mathbb{R}, V)$ . Then  $\varphi_{-\tau} \in AA(\mathbb{R}, V)$ . Now, by the composition Proposition 2.2, we have that  $\psi_\varphi(\cdot) = g(\cdot, \varphi_{-\tau}(\cdot)) \in AA(\mathbb{R}, V)$ . Moreover, applying similar arguments to that used in Theorem (3.7), we deduce that the Green operator

$$(\Gamma\varphi)(t) = \varphi_0(t) + \int_{\mathbb{R}} G(t, s)\psi_\varphi(s)ds$$

maps  $\Delta$  into  $\Delta$  and additionally we can prove that  $\Gamma : \Delta \rightarrow \Delta$  is a strict contraction. Thus, the Banach principle insures the existence of a unique  $\varphi \in \Delta$  satisfying (3.32). The proof is now complete.  $\square$

**3.3. Results for (1.1).** In this subsection we generalize the Theorems 3.7 and 3.8 to the case of the delay equation (1.1).

**Theorem 3.9.** *Consider that  $\tau > 0$  is a constant delay. Then, we have that the following assertions are valid:*

- (i) *If  $A, f$  and  $g$  satisfy the hypothesis of Theorem 3.7. Then, the conclusions of Theorem 3.7 holds for the delayed equation (1.1).*
- (ii) *If  $A, f$  and  $g$  satisfy the hypothesis of Theorem 3.8. Then, the conclusions of Theorem 3.8 are true for the delayed equation 1.1*

*Proof.* The proof follows by application of Theorems 3.7 and 3.8 since by Proposition 2.2 we have that  $g(\cdot, \phi(\cdot - \tau)) \in AA(\mathbb{R}, V)$  for every  $\phi \in AA(\mathbb{R}, \Delta(\varphi_0, \rho))$ . Indeed, this fact is a consequence of the following fact:  $\phi \in AA(\mathbb{R}, V)$  implies that the translation  $\phi_\tau(\cdot) = \phi(\cdot - \tau)$  is belongs  $AA(\mathbb{R}, V)$ .  $\square$

#### 4. APPLICATION TO THE DELAYED LASOTA-WAZEWSKA MODEL

The Lasota-Wazewska model is an autonomous differential equation of the form

$$y'(t) = -\delta y(t) + p e^{-\gamma y(t-\tau)}, \quad t \geq 0. \quad (4.1)$$

It was occupied by Wazewska-Czyzewska and Lasota [36] to describe the survival of red blood cells in the blood of an animal. In this equation,  $y(t)$  describes the number of red cells bloods in the time  $t$ ,  $\delta > 0$  is the probability of death of a red blood cell;  $p, \gamma$  are positive constants related with the production of red blood cells by unity of time and  $\tau$  is the time required to produce a red blood cell.

In this section, we study the following delayed model:

$$y'(t) = -\delta(t)y(t) + p(t)g(y(t-\tau)), \quad (4.2)$$

where  $\tau > 0$ ,  $\delta(\cdot)$ ,  $p(\cdot)$  are positive almost automorphic functions and  $g(\cdot)$  is a positive Lipschitz function with Lipschitz constant  $\gamma$ . Equation (4.2) models several situations in the real life, see [22].

We will assume the following condition

(D) The mean of  $\delta$  satisfies  $M(\delta) > \delta_- > 0$ .

In this section, the principal goal is the following Theorem:

**Theorem 4.1.** *In the above conditions, for  $\gamma$  sufficiently small, the equation (4.1) has a unique almost automorphic solution.*

By Lemma 2.3, the linear part of equation (4.1) has an exponential dichotomy. Let  $\psi(t)$  be a real almost automorphic function and consider the equation

$$y'(t) = -\delta(t)y(t) + p(t)g(\psi(t - \tau)). \quad (4.3)$$

Then, the bounded solution for the equation (4.3) satisfies

$$y(t) = \int_{-\infty}^t \exp\left(-\int_u^t \delta(s)ds\right) p(u)g(\psi(u - \tau))du.$$

The homogeneous part of equation of (4.3) has an exponential dichotomy and since  $\delta$  is almost automorphic function, by Lemma 2.8, it is integrally Bi-almost automorphic. Therefore, Theorem 4.1 follows from Theorem 3.7.

Taking  $g(x) = e^{-\gamma x}$ ,  $\alpha > 0$ , we have the Lasota-Ważewska model:

$$y'(t) = -\delta(t)y(t) + p(t)e^{-\gamma y(t-\tau)}, \quad t \geq 0. \quad (4.4)$$

**Corollary 4.2.** *For  $\gamma$  small enough, the delayed Lasota-Ważewska model (4.4) has a unique asymptotically stable almost automorphic solution.*

## REFERENCES

- [1] J. Blot, G. Mophou, G. M. N'Guérékata and D. Pennequin. Weighted pseudo almost automorphic functions and applications to abstract differential equations. *Nonlinear Analysis*, 71 (2009), 903-909.
- [2] S. Bochner. A new approach to almost periodicity. *Proceedings of the National Academic Science of the United States of America*, 48 (1962) 2039-2043.
- [3] S. Bochner. Continuous mapping of almost automorphic and almost periodic functions, *Proceedings of the National Academic Science of the United States of America*, 52 (1964) 907-910.
- [4] T. Caraballo, D. N. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I and II, *J. Differential Equations* 246(1) (2009), 108-128 and 1164-1186.
- [5] S. Castillo, M. Pinto, Dichotomy and almost automorphic solution of difference system. *Electron. J. Qual. Theory Differ. Equ.* 2013, No. 32, 17 pp.
- [6] A. Chávez, S. Castillo, M. Pinto, Discontinuous almost automorphic functions and almost automorphic solutions of differential equations with piecewise constant arguments. *Electron. J. Differential Equations* 2014, No. 56, 13 pp.
- [7] P. Cieutat, S. Fatajou and G. M. N'Guérékata. Composition of pseudo almost periodic and pseudo almost automorphic functions and applications to evolution equations. *Applicable Analysis*, 89 1 (2010), 11-27.
- [8] C. Corduneanu, *Almost Periodic Functions*. John Wiley and Sons, New York, 1968.
- [9] C. Cuevas, M. Pinto, Existence and uniqueness of pseudo-almost periodic solutions of semilinear Cauchy problems with non dense domain. *Nonlinear Anal.*, 45 (2001), no.1, 73-83.
- [10] H.-S. Ding, T.-J. Xiao, J. Liang, Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions. *J. Math. Anal. Appl.* 338 (2008) 141-151.
- [11] T. Diagana, *Pseudo Almost Periodic Functions in Banach Space*. Nova Sciences Publishers Inc., New York. 2007
- [12] S. Fatajou, N. V. Minh, G. N'Guérékata and A. Pankov. Stepanov-like almost automorphic solutions for nonautonomous evolution equations. *Electronic Journal of Differential Equations*, 121 (2007) 1-17.
- [13] C. Feng, On the existence and uniqueness of almost periodic solutions for delay logistic equations. *Appl. Math. Comput.*, 136(2-3) (2003) 478-494.
- [14] S. G. Gal and G. M. N'Guérékata. Almost automorphic fuzzy-number-valued functions. *Journal of Fuzzy Mathematics*, 13 1 (2005) 185-208.
- [15] J. A. Goldstein, G. M. N'Guérékata. Almost automorphic solutions of semilinear evolution equations. *Proc. Amer. Math. Soc.*, 133(8) (2005) 2401-2408.



- [16] R. C. Grimmer, Resolvent operators for integral equations in a Banach space. *Trans. Amer. Math. Soc.*, 273(1982) 333-349.
- [17] E. Hernández, J. P. C. dos Santos, Asymptotically almost periodic solutions for a class of partial integrodifferential equations. *Electron. J. Differential Equations*, 38 (2006), 1-8.
- [18] H. R. Henríquez, M. Pierri, P. Táboas, On S-asymptotically  $\omega$ -periodic functions on Banach spaces and applications, *J. Math. Anal. Appl.*, 343(2) (2008) 1119-1130.
- [19] H. R. Henríquez, M. Pierri, P. Táboas, Existence of S-asymptotically  $\omega$ -periodic solutions for abstract neutral equations. *Bull. Austral. Math. Soc.*, 78 (03) (2008), 365-382.
- [20] Z. C. Liang, Asymptotically periodic solutions of a class of second order nonlinear differential equations. *Proc. Amer. Math. Soc.*, 99(4) (1987), 693-699.
- [21] J. Liu, G. M. N'Guérékata and N. V. Minh. A Massera type theorem for almost automorphic solutions of differential equations. *Journal of Mathematical Analysis and Applications*, 299 (2004) 587-599.
- [22] E. Liz, C. Martínez, S. Trofimchuk, Attractivity properties of infinite delay Mackey-Glass type equations, *Differential and Integral Equations*, Vol. 15 (2002), 875-869.
- [23] N. V. Minh, T. Naito and G. N'Guérékata. A spectral countability condition for almost automorphy of solutions of differential equations. *Proceedings of the American Mathematical Society*, 134 11 (2006) 3257-3266.
- [24] N. V. Minh and T. T. Dat. On the almost automorphy of bounded solutions of differential equations with piecewise constant argument. *Journal of Mathematical Analysis and Applications*, 326 1 (2007) 165-178.
- [25] G. N'Guérékata. *Topics in Almost Automorphy*. Springer-Verlag, New York, 2005.
- [26] G. N'Guérékata. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*. Kluwer Academic/Plenum Publishers, New York, USA, 2001.
- [27] S. Nicola, M. Pierre, A note on S-asymptotically periodic functions. *Nonlinear Analysis, Real World Applications*, 10 (5)(2008), 2937-2938
- [28] M. Pinto, Pseudo-almost periodic solutions of neutral integral and differential equations and applications. *Nonlinear Analysis*, 72 (2010) 4377-4383.
- [29] M. Pinto, G. Robledo, Cauchy matrix for linear almost periodic systems and some consequences. *Nonlinear Anal.* 74 (2011), no. 16, 5426-5439.
- [30] M. Pinto, G. Robledo, Diagonalizability of nonautonomous linear systems with bounded continuous coefficients. *J. Math. Anal. Appl.* 407 (2013), no. 2, 513-526.
- [31] M. Pinto, G. Robledo, Existence and stability of almost periodic solutions in impulsive neural network models. *Appl. Math. Comput.* 217 (2010), no. 8, 4167-4177.
- [32] M. Pinto, V. Torres, G. Robledo, Asymptotic equivalence of almost periodic solutions for a class of perturbed almost periodic systems. *Glasg. Math. J.* 52 (2010), no. 3, 583-592.
- [33] W. R. Utz, P. Waltman, Asymptotically almost periodicity of solutions of a system of differential equations. *Proc. Amer. Math. Soc.*, (18)(1967), 597-601.
- [34] W. A. Veech. Almost automorphic functions. *Proceedings of the National Academy of Science of the United States of America*, 49 (1963) 462-464.
- [35] W. A. Veech. Almost automorphic function on groups. *American Journal of Mathematics*, 87 (1965) 719-751.
- [36] M. Wazewska-Czyżewska and A. Lasota, Mathematical problems of the red-blood cell system, *Ann. Polish Math. Soc. Ser. III, Appl. Math.* 6 (1976) 23-40.
- [37] F. Wei, K. Wang, Global stability and asymptotically periodic solutions for non autonomous cooperative Lotka-Volterra diffusion system. *Applied Math. and Computation*, 182(2006), 161-165.
- [38] F. Wei, K. Wang, Asymptotically periodic solutions of N-species cooperation system with time delay. *Nonlinear Analysis, Real World and Applications*. 7(2006), 591-596.
- [39] T. Xiao, X. Zhu, J. Liang, Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications, *Nonlinear Analysis*, 70(2009) 11, 4079-4085
- [40] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*. Applied Mathematical Sciences, 14, Springer-Verlag, New York- Heidelberg, 1975.
- [41] H. Zhao, Existence and global attractivity of almost periodic solutions for cellular neural network with distributed delays. *Appl. Math. Comput.*, 154(3)(2004), 683-695.
- [42] S. Zaidman. Almost automorphic solutions of same abstract evolution equations. *Istituto Lombardo, Accademia di Scienze e Lettere*, 110 2 (1976) 578-588.
- [43] S. Zaidman. Existence of asymptotically almost-periodic and of almost automorphic solutions for same classes of abstract differential equations. *Annales des Science Mathématiques du Québec*, 131 (1989) 79-88.
- [44] M. Zaki. Almost automorphic solutions of certain abstract differential equations. *Annali di Matematica pura et Applicata*, 101 1 (1974) 91-114.
- [45] C. Zhang. *Almost Periodic Type Functions and Ergodicity*. Science Press, Beijing, 2003.

ANÍBAL CORONEL

GMA, DEPARTAMENTO DE CIENCIAS BÁSICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DEL BÍO-BÍO, CAMPUS FERNANDO MAY, CHILLÁN, CHILE.

E-mail address: `acoronel@ubiobio.cl`



CHRISTOPHER MAULÉN

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CHILE  
*E-mail address:* `bleick@ug.uchile.cl`

MANUEL PINTO

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CHILE  
*E-mail address:* `pintoj.uchile@gmail.com`

DANIEL SEPULVEDA

ESCUELA DE MATEMÁTICAS Y ESTADÍSTICAS, UNIVERSIDAD CENTRAL DE CHILE  
*E-mail address:* `daniel.sep.oe@gmail.com`